3741. [2012 : 194, 196] *Proposed by Péter Ivády, Budapest, Hungary.*

Find the largest value of a and the smallest value of b for which the inequalities

$$\frac{ax}{a+x^2} < \sin x < \frac{bx}{b+x^2},$$

hold for all $0 < x < \frac{\pi}{2}$.

Composite of similar solutions by Arkady Alt, San Jose, CA, USA; and Kee-Wai Lau, Hong Kong, China.

We show that $a = \dfrac{\pi^2}{2(\pi-2)}$ and $b = 6$.

By simple computations, it is easy to show that the given inequalities are equivalent to

$$a < \frac{x^2 \sin x}{x - \sin x} < b. \tag{1}$$

To find the largest value of a and the smallest value of b for which (1) holds for $0 < x < \dfrac{\pi}{2}$, we let

$$f(x) = \frac{x^2 \sin x}{x - \sin x}, \quad x \in \left(0, \frac{\pi}{2}\right).$$

Then

$$\begin{aligned} f'(x) &= \frac{1}{(x-\sin x)^2}\Big((x-\sin x)(2x\sin x + x^2\cos x) - (x^2 \sin x)(1-\cos x)\Big) \\ &= \frac{1}{(x-\sin x)^2}(x^2 \sin x + x^3 \cos x - 2x\sin^2 x) = \frac{xg(x)}{(x-\sin x)^2} \end{aligned} \tag{2}$$

where $g(x) = x\sin x + x^2 \cos x - 2\sin^2 x$.

Since

$$0 < x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}$$

and

$$\cos x < 1 - \frac{x^2}{2} + \frac{x^4}{24},$$

we have

$$\begin{aligned} g(x) &< x\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) + x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) - 2\left(x - \frac{x^3}{6}\right)^2 \\ &= \left(2x^2 - \frac{2}{3}x^4 + \frac{1}{20}x^6\right) - \left(2x^2 - \frac{2}{3}x^4 + \frac{1}{18}x^6\right) = -\frac{1}{180}x^6 < 0. \end{aligned} \tag{3}$$

From (2) and (3) we have $f'(x) < 0$ so $f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ which implies

$$\lim_{x \to \frac{\pi}{2}^-} f(x) < f(x) < \lim_{x \to 0^+} f(x). \tag{4}$$

Since

$$\lim_{x\to\frac{\pi}{2}^-} f(x) = \frac{\left(\frac{\pi}{2}\right)^2}{\frac{\pi}{2}-1} = \frac{\pi^2}{2(\pi-2)}$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x\sin x}{1-\frac{\sin x}{x}} = \lim_{x\to 0^+} \frac{x\left(x-\frac{x^3}{3!}+\cdots\right)}{1-\left(1-\frac{x^2}{3!}+\cdots\right)}$$
$$= \lim_{x\to 0^+} \frac{x^2+o(x^3)}{\frac{x^2}{6}+o(x^3)} = 6,$$

we have from (4) that $\frac{\pi^2}{2(\pi-2)} < f(x) < 6$ which completes the proof.

Also solved by JOHN HAWKINS and DAVID R. STONE, Georgia Southern University, Statesboro, GA, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; PAOLO PERFETTI, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Rome, Italy; ALBERT STADLER, Herrliberg, Switzerland; MIHAÏ-IOAN STOËNESCU, Bischwiller, France; and the proposer. There was also an incorrect solution.

3742. [2012 : 194, 196] *Proposed by Michel Bataille, Rouen, France.*

In a scalene triangle ABC, let K, L, M be the feet of the altitudes from A, B, C, and P, Q, R be the midpoints of BC, CA, AB, respectively. Let LM and QR intersect at X, MK and RP at Y, KL and PQ at Z. Show that AX, BY, CZ are parallel.

Solution by Ricardo Barroso Campos, University of Seville, Seville, Spain.

Since $BMLC$ is a cyclic quadrilateral, $\angle MLA = \angle ABC = \angle ARQ$. Therefore $LQRM$ is cyclic and $XQ \cdot XR = XL \cdot XM$.

Let Γ be the circumcircle of ARQ. Since ABC maps to ARQ by a dilatation with factor $\frac{1}{2}$, its centre is the midpoint S of the segment AO, where O is the circumcentre of triangle ABC. Let AX intersect this circle at $U \neq A$. Then

$$XQ \cdot XR = XU \cdot XA.$$

Let Δ be the circle whose diameter is AH. Since AH subtends right angles at L and M, this circle is the circumcircle of ALM with diameter AH. Accordingly, its centre is the midpoint T of AH. Let AX intersect this circle at $V \neq A$. Then

$$XL \cdot XM = XV \cdot XA.$$

Since $XU \cdot XA = XQ \cdot XR = XL \cdot XM = XV \cdot XA$, $U = V$. Thus, AU is a chord of both Γ and Δ, so its right bisector contains both the centres S and T. But ST is the image of OH under a dilatation with centre A, so $ST \| OH$. It follows that $AX \perp OH$. Similarly it can be shown that both BY and CZ are perpendicular to OH, from which the desired result follows.